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Normal form approach to and nonlinear optics analysis of the IOTA ring

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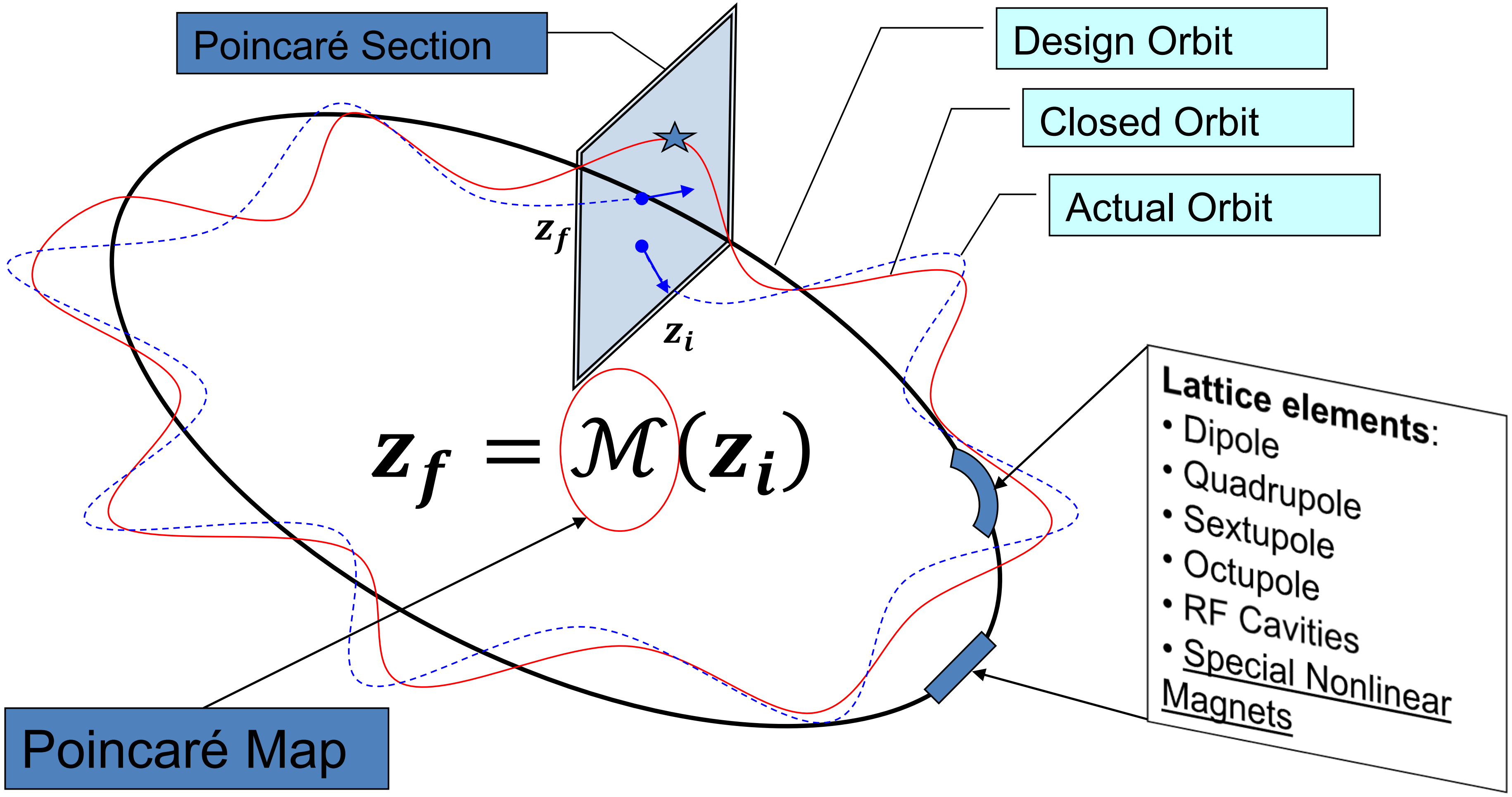
Outline

- Key concepts:
  - **Integrability**: Hamiltonians and Symplectic Maps
  - (Birkhoff) **Normal Form**
- The **relation between** integrability and normal form
- Integrable Hamiltonians of interest:
  - **Polynomial** potentials
  - **Non-polynomial** potential: **IOTA**
- Normal Form applied to IOTA: its **computation**, **limitations** and some **results**

Mathematical Model of Periodic Accelerators



Integrability

A (time-periodic) Hamiltonian (vector field) X_H is called completely integrable if, for any fixed time, there are sufficiently many (half the phase space dimensionality) functionally independent analytical functions f_i in involution (Poisson commuting).

$$\{f_i, f_j\} = 0, \qquad i, j = 1, \dots, n.$$

A symplectic map $\mathcal{M}$ is called completely integrable if there are sufficiently many (half the phase space dimensionality) functionally independent analytical invariants f_i in involution (Poisson commuting).

$$f_i \circ \mathcal{M} = f_i, \qquad i = 1, \dots, n.$$

Hamiltonian and Map Integrability Relationship

The one-turn map of a time-periodic integrable Hamiltonian H is an integrable symplectic map $\mathcal{M}$.

If a symplectic map $\mathcal{M}$ is integrable, then it is the one-turn map of a time-periodic integrable Hamiltonian H.

Normal Form

Hamiltonian Picture: Action-Angle Variables

$$H(\boldsymbol{q},\boldsymbol{p}) \mapsto K(\boldsymbol{J}), \;\; J_i = \oint p_i \mathrm{d}q_i \,, \quad w_i = \frac{\partial K}{\partial J_i} t$$

Map Picture: Special Symplectic Change of Variables

$$\mathcal{M} = \mathcal{L} + \mathcal{H} \mapsto \mathcal{N} \text{ such that } \hat{\mathcal{L}} \circ \mathcal{N} = \mathcal{N} \circ \hat{\mathcal{L}}$$

$\hat{\mathcal{L}}$ – real symplectic Jordan canonical form of $\mathcal{L}$

Convergence of the Normalizing Transformation

There exists a real-analytic, symplectic Normal Form transformation if and only if the Hamiltonian/Symplectic Map is integrable.

H. Ito (1997)

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{A}} \circ \boldsymbol{\mathcal{N}} \circ \boldsymbol{\mathcal{A}}^{-1}$$

$\mathcal{N}$ – normal form of $\mathcal{M}$

$\mathcal{A}$ – normalizing transformation

$f_i \circ \mathcal{A}$ – invariants of $\mathcal{N}$

Hamiltonians of Interest

$$H\left(x,a,y,b,\delta;s\right)=-\sqrt{(1+ \ldots\; a_y\left(x,y,s\right))^2}-a_s\left(x,y,s\right)+\delta$$

$$H=\frac{1}{2}(p_x^2+p_y^2)+V(x,y)$$

Polynomial

$$V(x,y)=V_{\min}(x,y)+\cdots+V_{\max}(x,y)$$

$$H=\frac{1}{2}(p_x^2+p_y^2)+V_{\min}(x,y)$$

$$H=\frac{1}{2}(p_x^2+p_y^2)+V_{\max}(x,y)$$

Non-Polynomial

Bertrand-Darboux: the potential must be separable in *elliptic coordinates* (or polar, or parabolic, or Cartesian)

Integrable Polynomial Potentials & The Laplacian

A list of all integrable 2D homogeneous polynomial potentials

- with a polynomial first integral linear in the momenta

$$V_k = (x^2 + y^2)^{k/2}, \quad k = \text{even}.$$

- with a polynomial first integral quadratic in the momenta

$$V_k = \frac{1}{r}\left[\left(\frac{r+x}{2}\right)^{k+1} + (-1)^k \left(\frac{r-x}{2}\right)^{k+1}\right], \quad V_k = Ax^k + By^k.$$

- with a polynomial first integral quartic in the momenta

$$V_3 = x^3 + \frac{3}{16}xy^2, \quad V_3 = x^3 + \frac{1}{2}xy^2 + \frac{\sqrt{3}\,i}{18}y^3, \quad V_4 = x^4 + \frac{3}{4}x^2y^2 + \frac{1}{8}y^4.$$

No one has discovered any polynomial first integral which is *genuinely* quintic or higher orders in the momenta. It is still an open problem whether or not there exist such polynomial first integrals.

$$\Delta V \neq 0$$

Except quadrupole

Integrable Non-Polynomial Potentials: IOTA

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^2 + q_2^2}{2} + \frac{f_2(\xi) + g_2(\eta)}{\xi^2 - \eta^2}$$

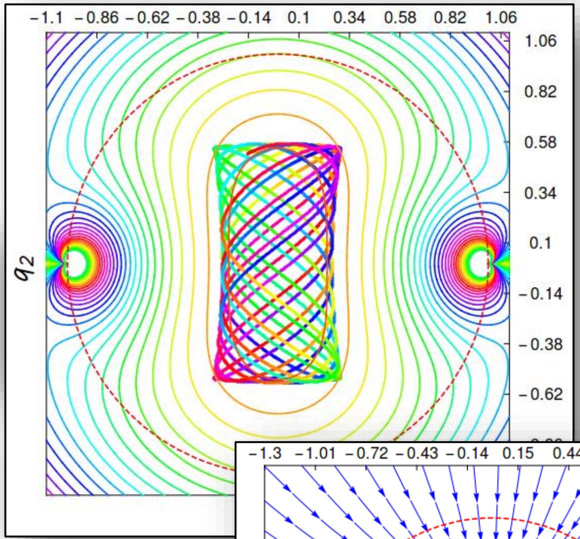
$$(\xi, \eta) = \left(\frac{\sqrt{(q_1 + c)^2 + q_2^2} + \sqrt{(q_1 - c)^2 + q_2^2}}{2c}, \frac{\sqrt{(q_1 + c)^2 + q_2^2} - \sqrt{(q_1 - c)^2 + q_2^2}}{2c} \right)$$

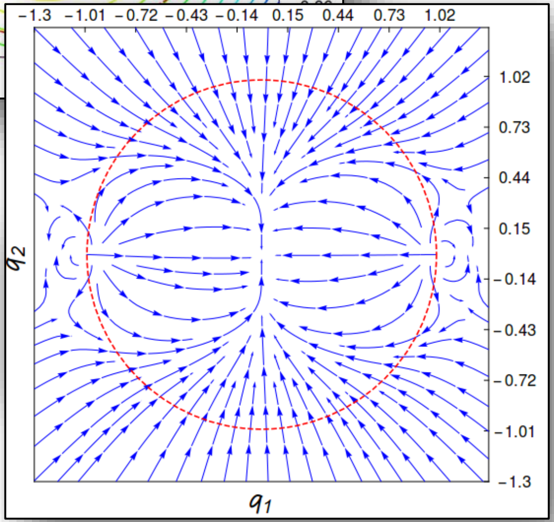
$$f_2(\xi) = -\xi\sqrt{\xi^2 - 1}(d + t\cosh^{-1}(\xi)) \qquad g_2(\eta) = -\eta\sqrt{1 - \eta^2}(b + t\cos^{-1}(\eta))$$

$$d = 0 \qquad b = -\pi t/2$$

$$\mathrm{V}(r, \varphi) = -\frac{t}{c^2}\left(r^2\cos(2\varphi) + \frac{2}{3c^2}r^4\cos(4\varphi) + \frac{8}{15c^4}r^6\cos(6\varphi) + \dots\right)$$

$$\boxed{\Delta V = 0}$$

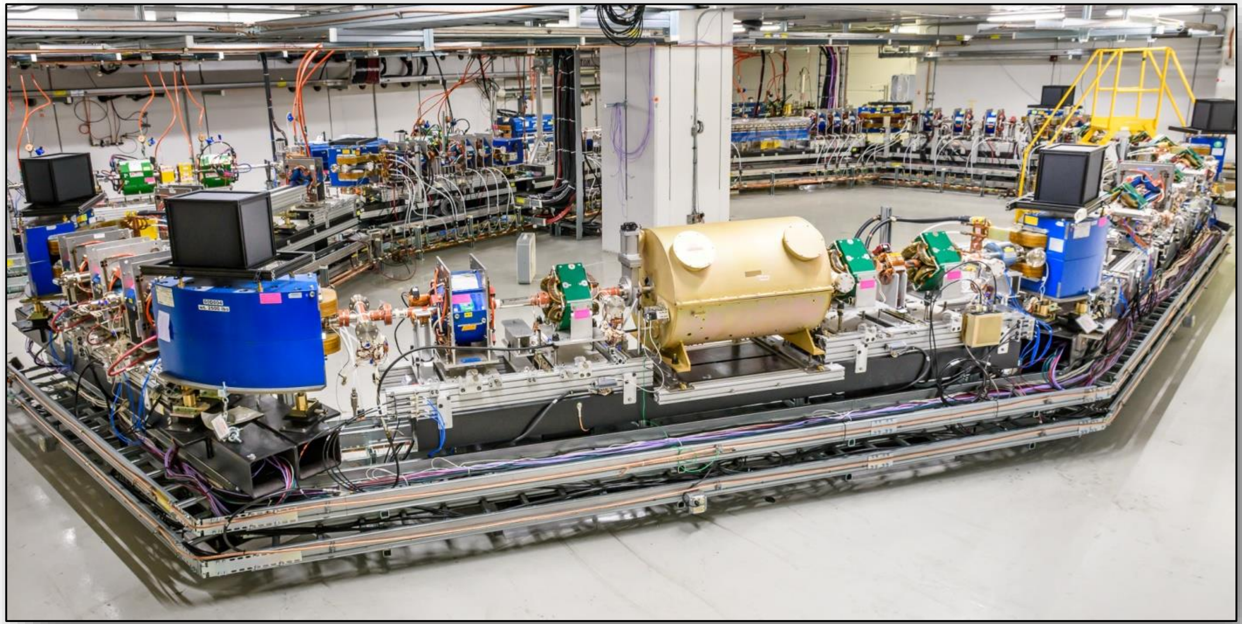




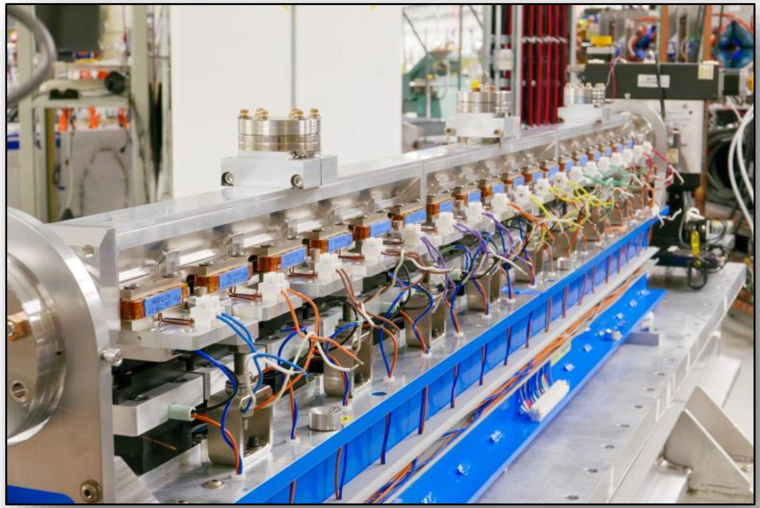
Nonlinear Dynamics of (Near-) Integrable Accelerators: IOTA

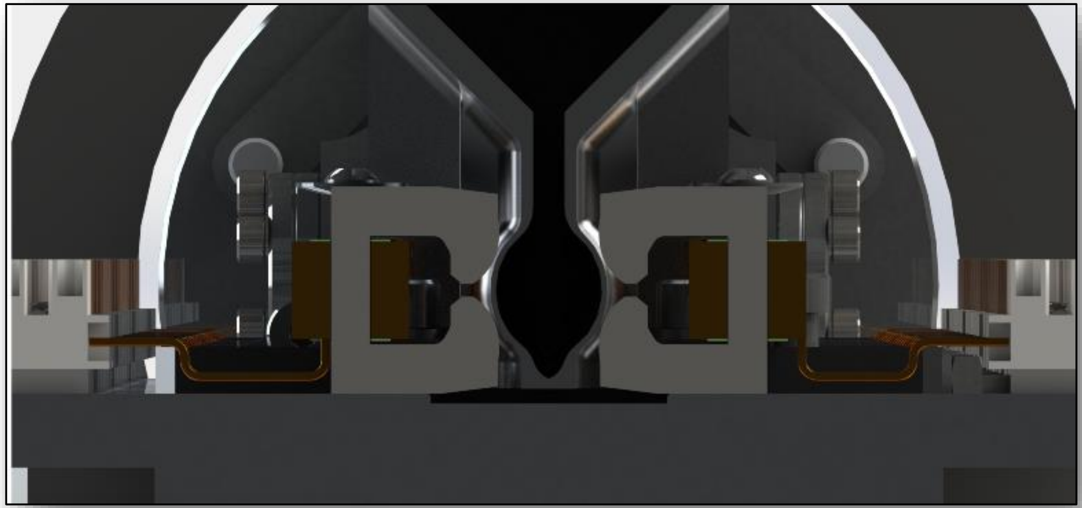
A new class of accelerators:

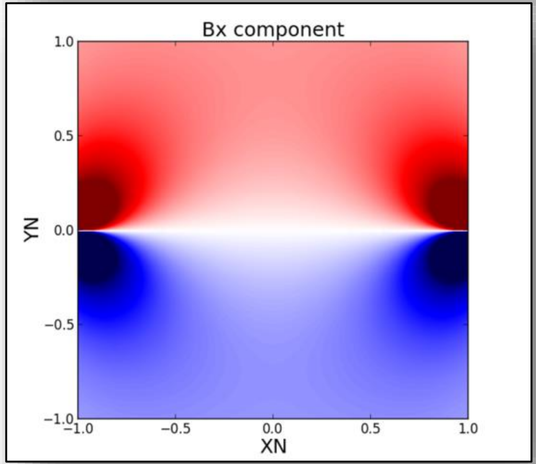
- *Special nonlinear magnetic fields* to spread out the circulating particles' oscillations
- The induced collective effects *damp any potential instabilities* (conservative relaxation is the beam's immune system)
- Challenge is to *maintain orbit stability* while this is happening
- *Impose (near-) integrability*, which implies absence of resonances and hence chaos
- Push the *intensity frontier* of beam physics

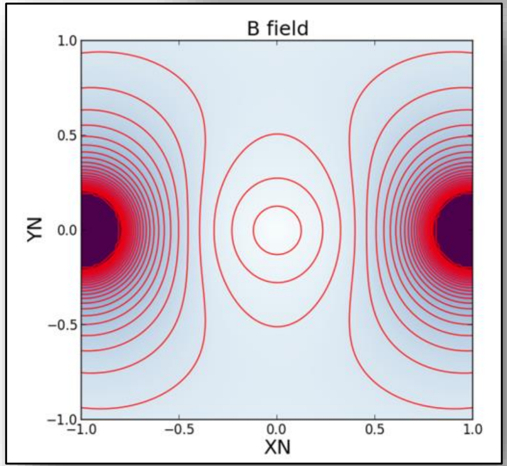


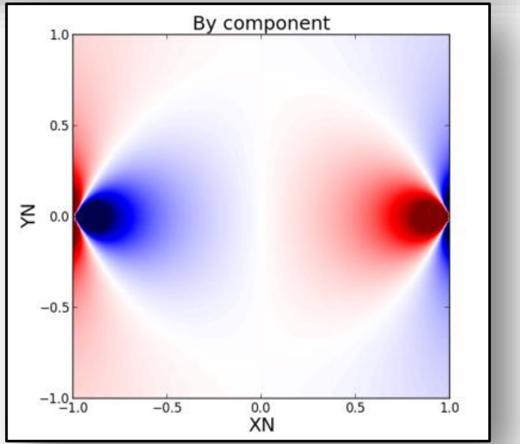
The IOTA Nonlinear Magnet











DA Methods for Map Extraction and Normal Form

Equations of the Motion

$$\frac{dx}{ds} = (1+hx)\,\frac{a}{\sqrt{(p/p_0)^2 - a^2 - b^2}}$$

$$\frac{dy}{ds} = (1+hx)\,\frac{b}{\sqrt{(p/p_0)^2 - a^2 - b^2}}$$

$$\frac{da}{ds} = h \cdot \sqrt{(p/p_0)^2 - a^2 - b^2} + \frac{e}{p_0}\left[\frac{1}{\dot{s}}E_x + \frac{dy}{ds}B_s - (1+hx)\,B_y\right]$$

$$\frac{db}{ds} = \frac{e}{p_0}\left[\frac{1}{\dot{s}}E_y - \frac{dx}{ds}B_s + (1+hx)\,B_x\right]$$

DA Propagator

$$\vec{z}_f = \sum_{i=1}^{\infty} \frac{t^i \cdot L^i_{\vec{f}}}{i!}\,\mathcal{I}$$

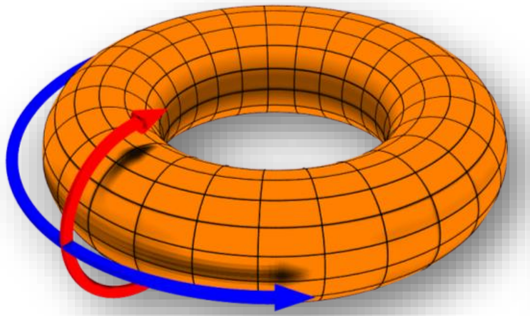
Transfer Map

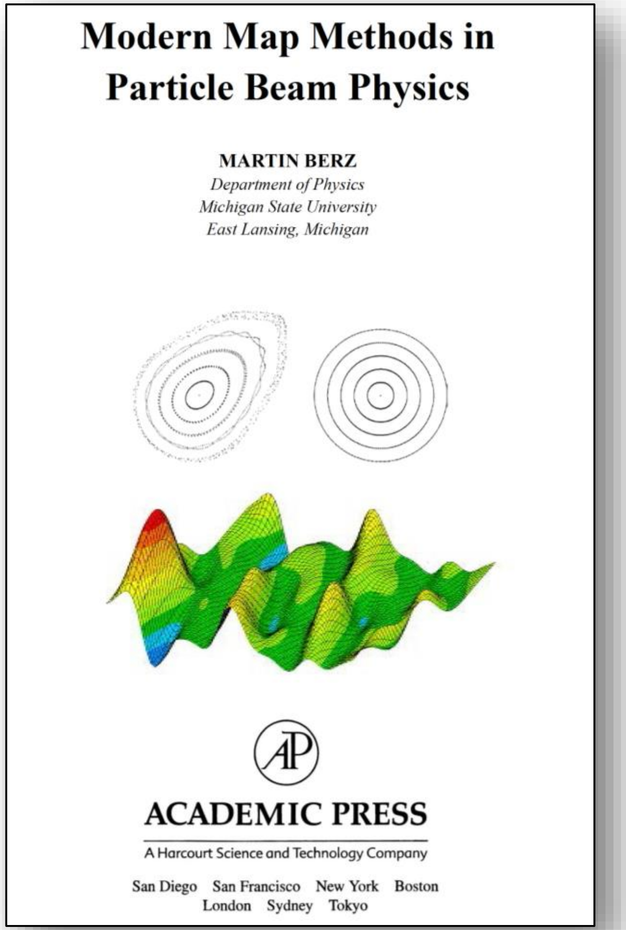
$$\vec{z}_f = \mathcal{M}(\vec{z}_i, \vec{\delta})$$

Normal Form

$$\mathcal{A} \circ \mathcal{M} \circ \mathcal{A}^{-1}$$

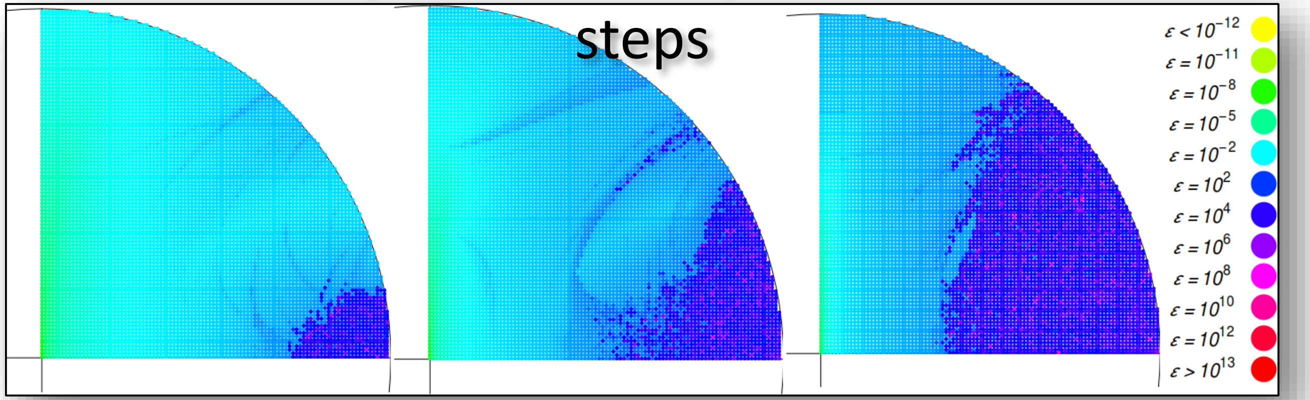
$$\mathcal{A}_m = \mathcal{I} + \mathcal{T}_m$$



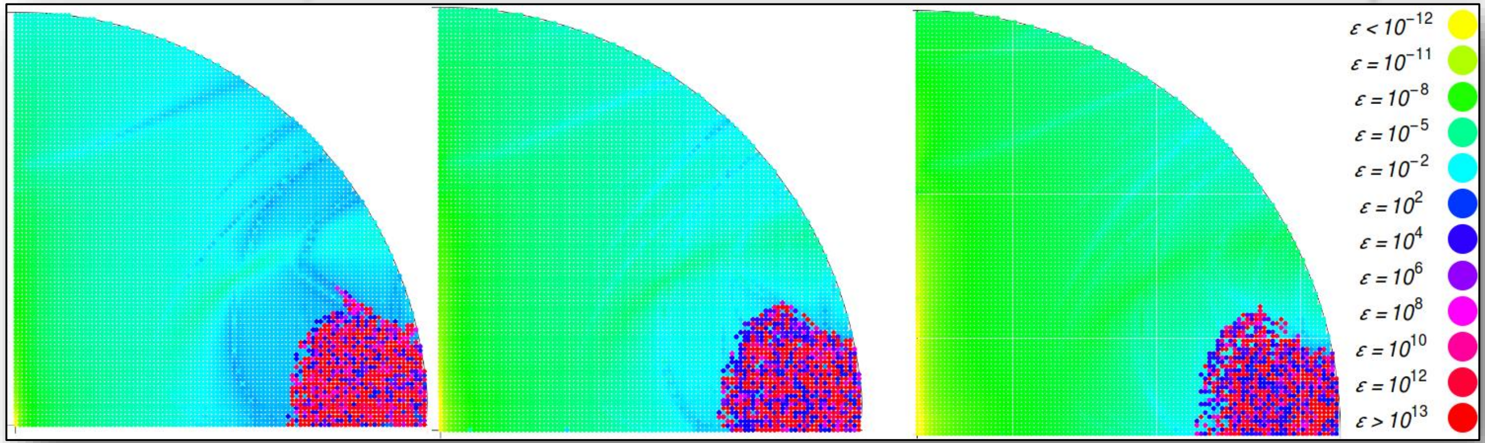


(Symplectic) Integrators Destroy Integrability

Stormer-Verlet (second order) at three different time steps



Lobatto3, Lobatto4 and Lobatto5 at same time steps



Depends on:

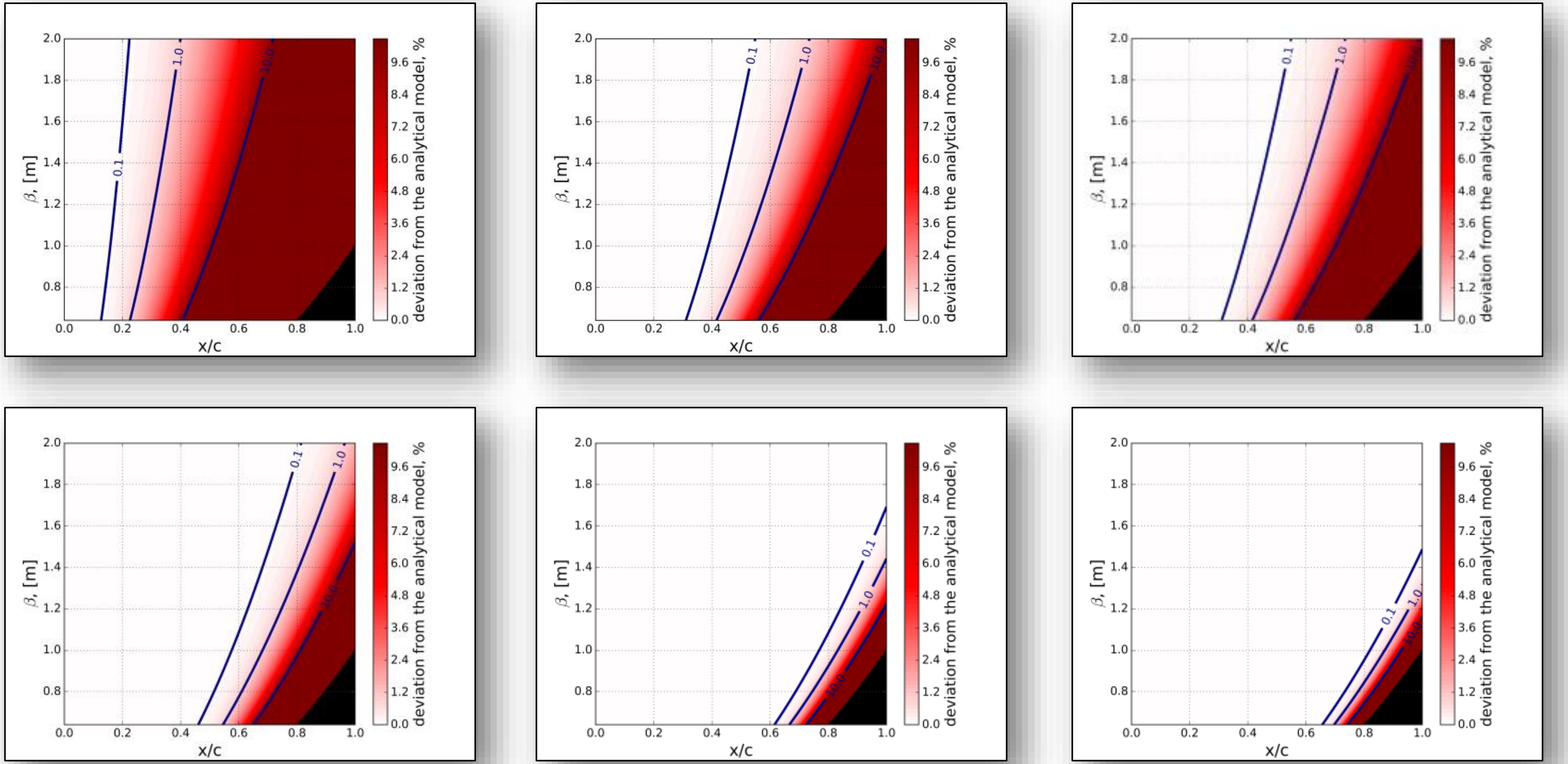
- Time step
- Order
- Method

Strengths and Limitations of Normal Form Analysis of IOTA

☺ Amplitude Dependent Tune Shifts
☺ Amplitude Dependent Chromaticities
☺ Resonance Strengths and Widths
☺ High-Order Matching

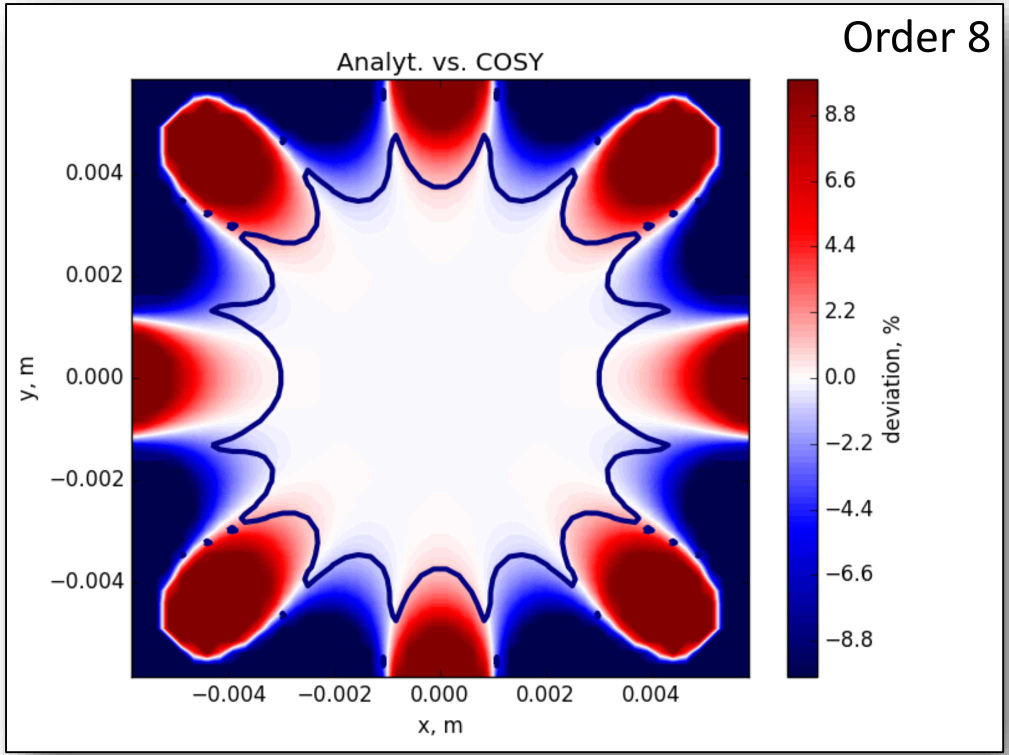
☹ Any numerical integrator will break integrability
☹ Approximating the nearly-integrable system by an integrable one using the normal form, in general will lead to a slightly different Hamiltonian system
☹ Slow convergence due to the non-polynomial potential
☹ Paradoxical: in principle, normal form should work best for IOTA (uniform convergence), but in practice it is somewhat limited (slow convergence), allowing comparison of expansion coefficients of different cases and small-to-medium amplitude analysis

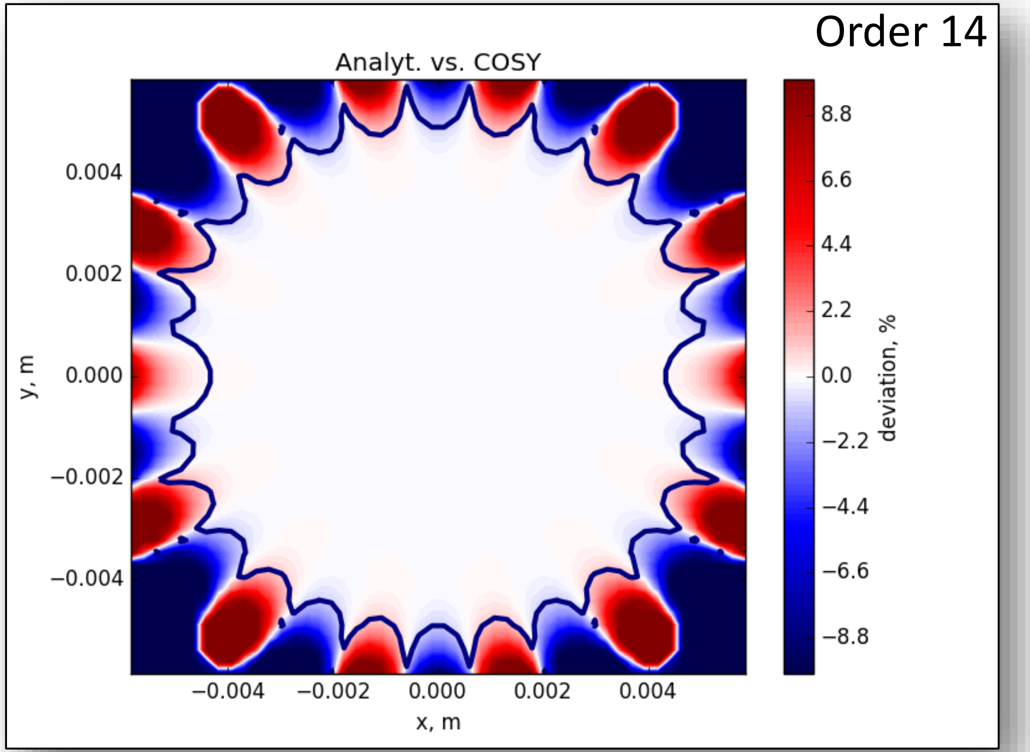
Taylor approximation of mid-plane B_y component



Taylor approximation of B_y component of the non-linear potential and it's deviation from the analytical model. From top left to bottom right Taylor series of order: 4, 8, 10, 14, 18, 30.

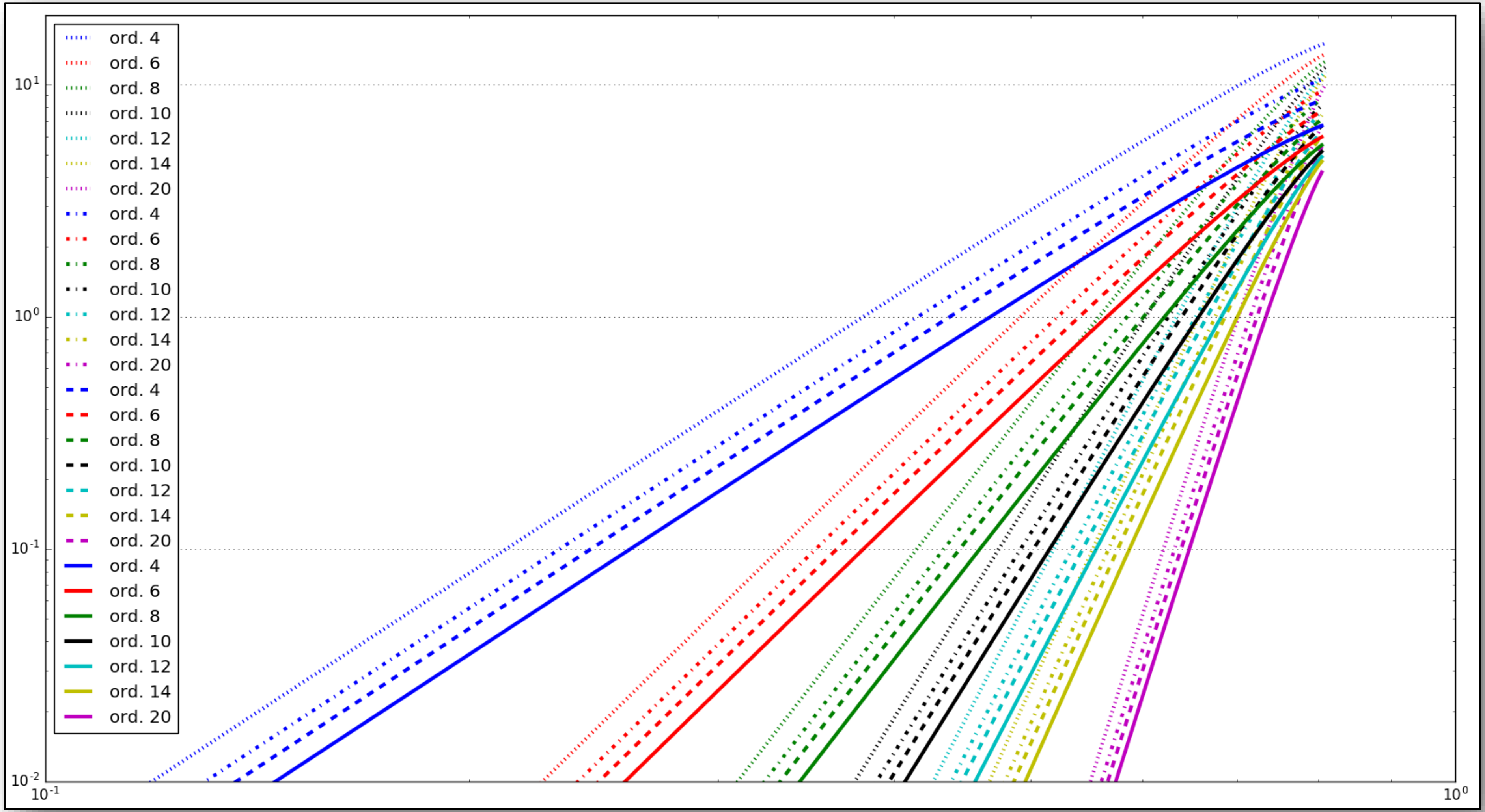
Taylor approximation of transverse field (magnet middle)





Solid line shows the level of 1% relative error

Integrated Field Error (%)



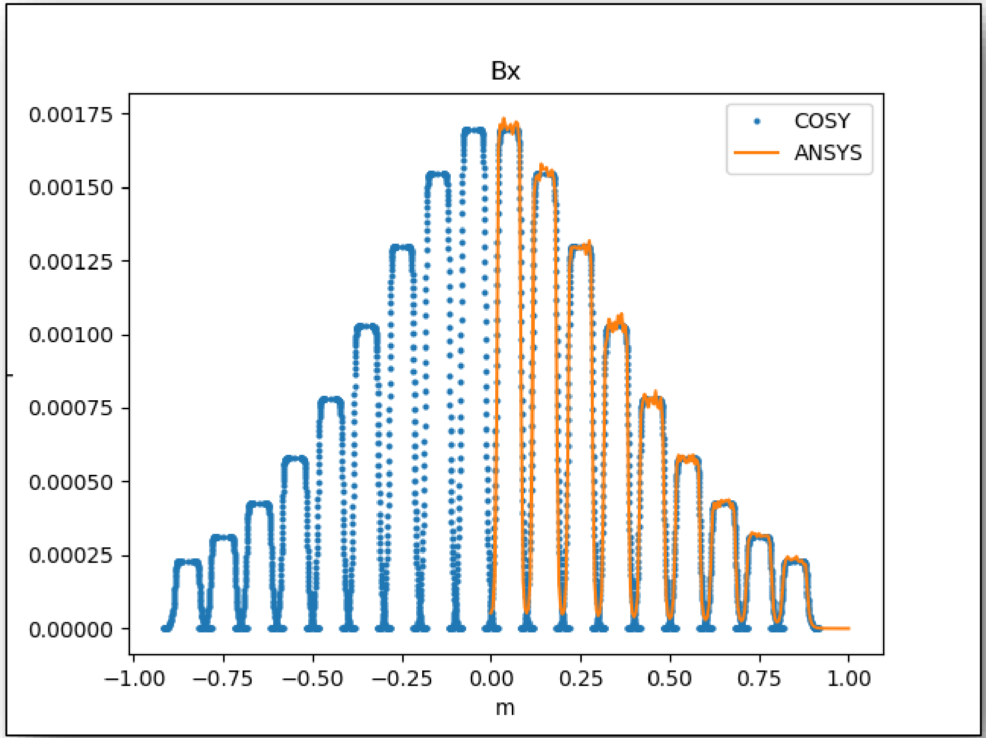
The Four Study Cases

Non-linear magnet types:

1. **Continuous**: the number of non-linear longitudinal slices is 1000. No drifts.
2. **Split**: the geometry of the real magnet (18 slices). Magnet includes drifts.

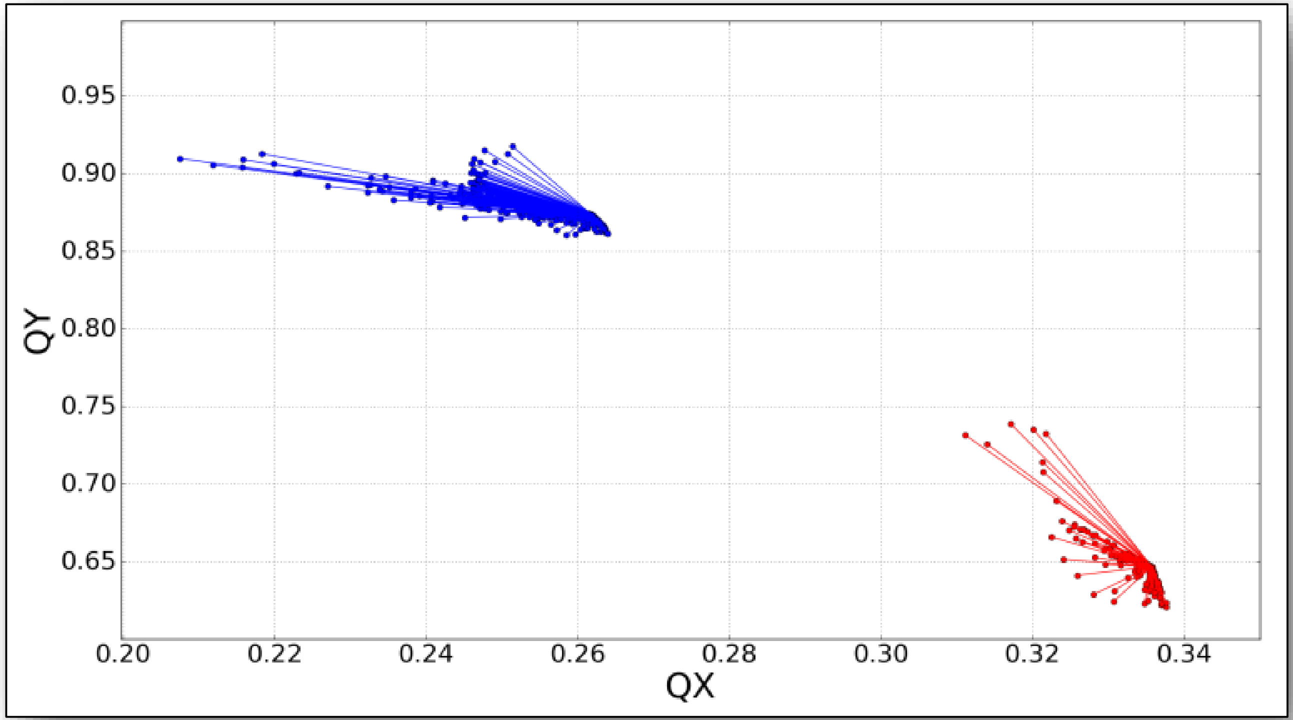
Lattices:

1. **Continuous:** Non-linear magnet + kick matrix.
2. **Split:** Non-linear magnet + kick matrix.
3. **2-Continuous:** Two non-linear magnets in the IOTA ring.
4. **2-Split:** Two non-linear magnets in the IOTA ring.



Observation point is the IOTA injection

Results: Tunes



Blue – split magnet: maximal tune spread in X ~0.06; Y ~0.06

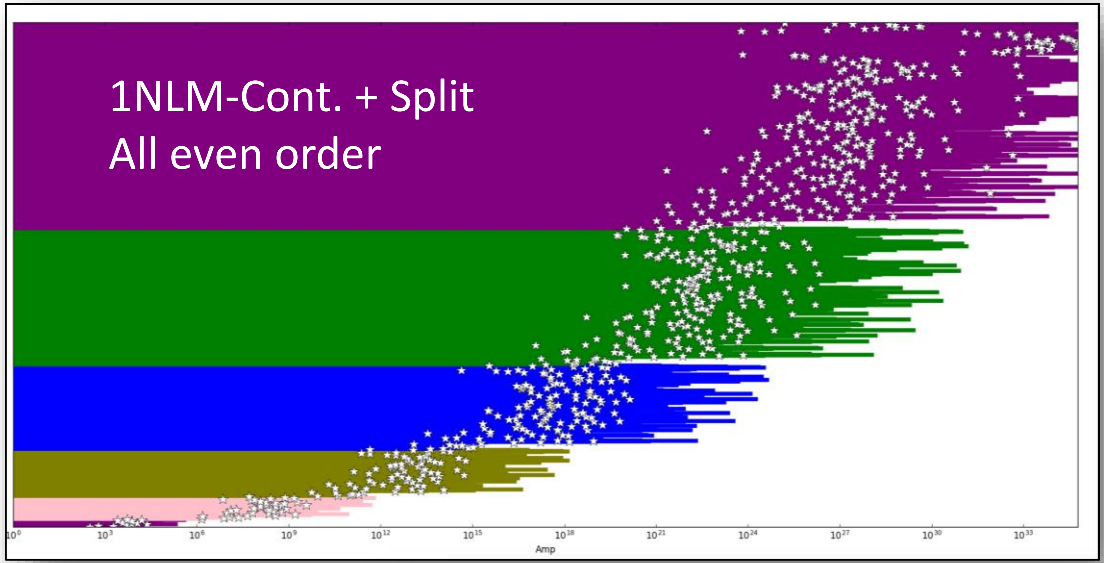
Red – continuous magnet: maximal tune spread in X ~0.03; Y ~0.12

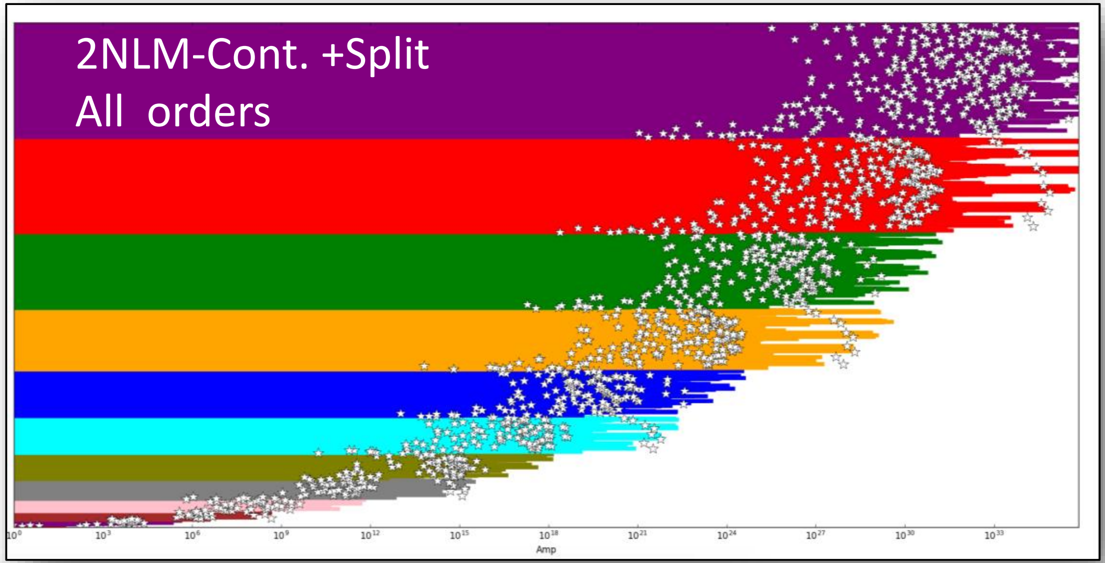
Results: Chromaticities (H & V)

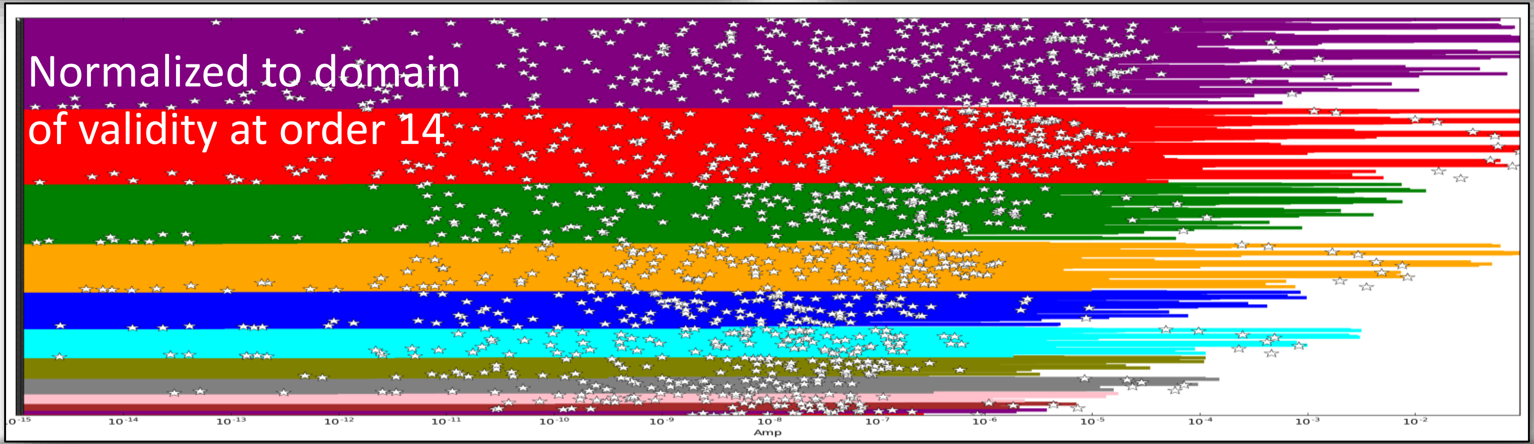
1	0.2617014558063252	0	0	0	0	0	0
2	-9.501058154667085	1	0	0	0	0	1
3	1431.974081436545	2	2	0	0	0	0
4	-5779.199946597837	2	0	0	2	0	0
5	38.44651165214852	2	0	0	0	0	2
6	2514.591360537211	3	2	0	0	0	1
7	-232287.3985768651	3	0	0	2	0	1
8	-612.4426449070812	3	0	0	0	0	3
9	10005243.38413168	4	4	0	0	0	0
10	-345163280.5689275	4	2	0	2	0	0
11	250015609.9222542	4	0	0	4	0	0
12	-116624.4818519582	4	2	0	0	0	2
13	-415915030.3996868	4	0	0	2	0	2
14	218923.8178066820	4	0	0	0	0	4
15	-17618539.87187787	5	4	0	0	0	1
16	-667401322667.9651	5	2	0	2	0	1
17	138359658721.3394	5	0	0	4	0	1
18	791633.9439375501	5	2	0	0	0	3
19	-1096403897439.739	5	0	0	2	0	3
20	11390970.70547216	5	0	0	0	0	5

1	0.8739656666948553	0	0	0	0	0	0
2	-8.017422621468725	1	0	0	0	0	1
3	-5779.19946597853	2	2	0	0	0	0
4	5816.704163812616	2	0	0	2	0	0
5	74.99183048522569	2	0	0	0	0	2
6	-232287.3985768599	3	2	0	0	0	1
7	64313.90358739401	3	0	0	2	0	1
8	-3138.602002283135	3	0	0	0	0	3
9	-172581640.2844621	4	4	0	0	0	0
10	500031219.8445172	4	2	0	2	0	0
11	-216502513.8884833	4	0	0	4	0	0
12	-415915030.3996801	4	2	0	0	0	2
13	-201602774.0703870	4	0	0	2	0	2
14	-175681.1251752207	4	0	0	0	0	4
15	-333700661333.9794	5	4	0	0	0	1
16	276719317442.7849	5	2	0	2	0	1
17	134406798131.9375	5	0	0	4	0	1
18	-1096403897439.712	5	2	0	0	0	3
19	-548773117164.5134	5	0	0	2	0	3
20	-49364641.60627148	5	0	0	0	0	5

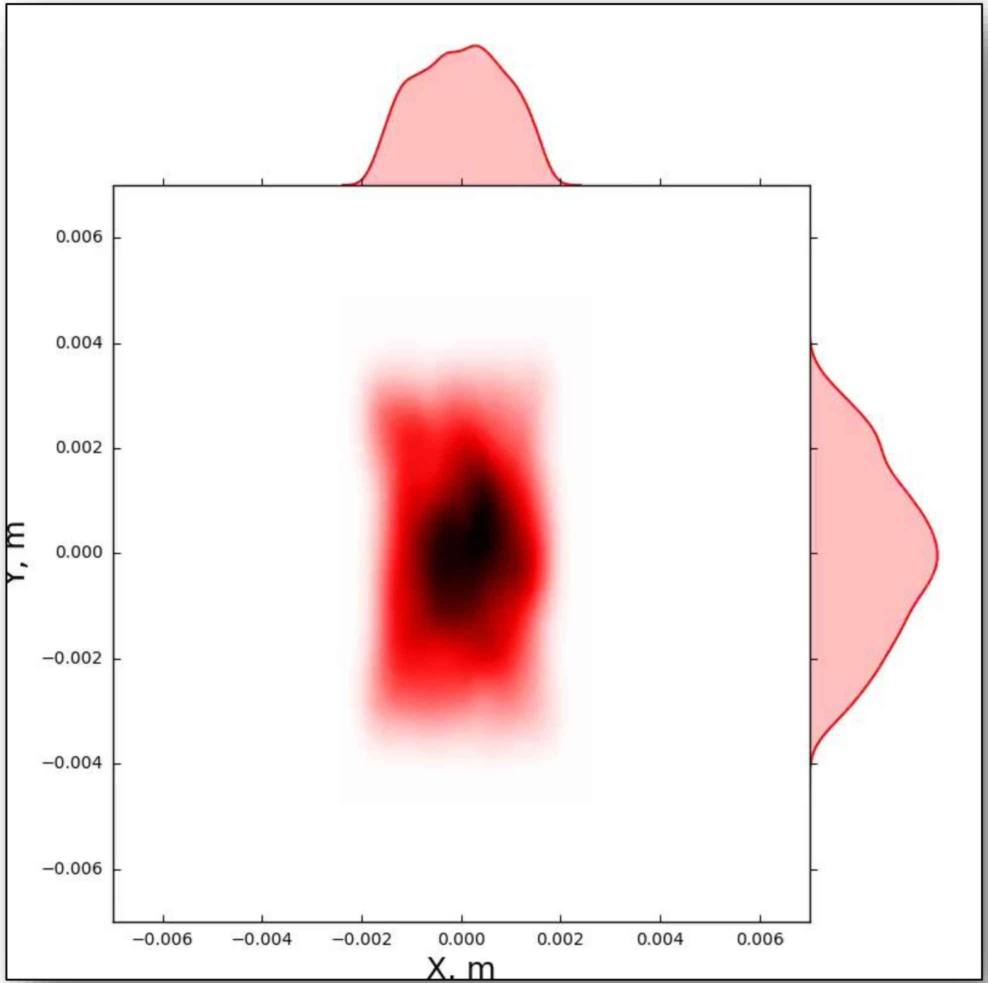
Results: Resonance Strengths







Order 14 Matching and Tracking



Summary and Conclusions

- Integrable Hamiltonians and Symplectic Maps are in one-to-one correspondence; map picture advantageous for complicated systems
- Normal From transformations converge uniformly
- For the IOTA nonlinear potential convergence is slow, in practice limiting the domain of applicability of normal forms
- Within their domain of applicability, normal forms show significant strengths in nonlinear phase space topology mapping and dynamics analysis:
  - Amplitude- and parameter-dependent tune shifts, spreads, footprints
  - Resonance strengths and widths
  - High order beam matching