Mean-field density evolution of bunched particles with non-zero initial velocity

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Outline

I. Literature review

II. New cylindrical expression and validation

III. Conclusions
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The pancake regime

Cigar (normal) : $\Delta z >> \Delta x$

- Short bunches
- Denser bunches
- Virtual cathode limit

Valfells, 2002 Physics of Plasmas

- Space-Charge dominated
  - Intense non-linearities
- Cost: Less overall particles

Pancake (UEM) : $\Delta z << \Delta x$
Density Projection

Slicing

Zerbe, 2018 PRAB
Aperture → Brightness increase

\[ \epsilon_x^2 = |\text{COV}(x, p_x)| \]

\[ B_{4D} = \frac{N}{\epsilon_x^2} \]

\[ B_{6D} = \frac{N}{\epsilon_x^2 \epsilon_z} \]

Williams, 2017 Presentation
Fluid models for this?

\[ F_e = -k^2 x \]

\[ \rho_0 = \text{initial probability-like density} \]

\[ \rho_u = \frac{k^2 m v^2}{4 \pi e^2} = \frac{N}{\text{Perveance} / k^2} \]

\[ \rho(r, z) = \frac{\rho_u}{1 + \left( \frac{\rho_u}{\rho_0} - 1 \right) \cos(kz)} \]

Formation of shock (simulations only – laminar theory does not see this phenomenon)


\( r @ \text{different t's} \)
Fluid models for this?

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Formation of shock (simulations only – laminar theory does not see this phenomenon)


Here \( k = 0 \)
Fluid models for this?

Evolution of pancake width – A 1D problem

\[ \rho(z; t) = \rho_0 \frac{1}{1 + \frac{q \sum_{\text{tot}}}{2 m \varepsilon_0} \rho_0 t^2} \]

Longitudinal

Time evolved instead of along trajectory

1D in figure

Fluid models for this?

Evolution of pancake width – A 1D problem

\[ \rho(z; t) = \frac{\rho_0}{1 + \frac{q \sum_{t_{\text{ot}}} \rho_0 t^2}{2 m \epsilon_0}} \]

Time evolved instead of along trajectory

Longitudinal

Attempt to model for dense cylindrical (cigar) beams to see if shock can be observed
Our fluid models

Cylindrical

\[ \rho(r; t) = \frac{\rho_0^2}{r^2} \frac{1}{1 + \left( \frac{\rho_0^2}{\rho_2} - 1 \right) 2 \sqrt{\ln \frac{r}{r_0}} F \left( \sqrt{\ln \frac{r}{r_0}} \right)} \]

Spherical

\[ \rho(r; t) = \frac{\rho_0^3}{r^3} \frac{1}{1 + \frac{3}{2} \left( \frac{\rho_0^3}{\rho_3} - 1 \right) \frac{r_0}{r} \sqrt{1 - \frac{r_0^2}{r^2} \tanh^{-1} \left( \sqrt{1 - \frac{r_0^2}{r^2}} + 1 - \frac{r_0}{r} \right)}} \]

Solid lines: Theory
Circles: PIC (Warp)
Triangles: N-particle (COSY Infinity and home-made)
Take aways

• Step away from UEM
  - Discuss cylindrical evolution

• PIC (Vlasov) and N-particle (Coulomb) typical
  - New self-consistent laminar density evolution
    • Validation
    • Identify when it breaks down

• Our previous fluid model has no velocity
  - I fixed that
Laminar beams

Wangler's Prin. RF Linac

The ideal beam with highest beam quality is called the laminar beam because it exhibits laminar-like flow. A laminar beam represents the ideal of a highly ordered and coherent beam, which is never exactly realized.
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Derivation idea

- Lagrangian particle with initial velocity can be mapped to Lagrangian particle at different time that has zero velocity
  - $v_{r2}$: velocity scale
    \[ v_{r2} = \sqrt{\frac{q \Lambda_{tot} \bar{\rho}_{02}}{m \epsilon}} r_0 \]
  - $r_{t2}$: turn-around radius
    \[ r_{t2} = r_0 e^{-v_0^2/v_{r2}^2} \]
  - $t_{ft2}$: time-position free expansion relation from turn-around point under spherical symmetry
  - $t_{t2}$: time to turn around point (- if $v_0 < 0$)

- Time position-relation can be summarized as
  \[ t = \pm t_{ft2} - t_{t2} \]

- Evolution derived exactly like previously
Cylindrically-symmetric uniform distribution with non-zero initial velocity

\[ v_0 = C \frac{r_0}{R} \]
\[ v_{r2} = 10^5 \frac{m}{s} \frac{r_0}{R} \]

\[ C = 10^5 \text{ m/s} \]
\[ C = -10^5 \text{ m/s} \]

Large outward
Large inward

Solid lines: Theory
Circles: PIC (Warp)
Analogous Gaussian

\[ v_0 = C \frac{r_0}{\sigma_r} \]
\[ v_{r2} = 1.4 \times 10^5 \frac{m}{s} \sqrt{1 - e^{-\frac{r_0^2}{2\sigma_r^2}}} \]

Solid lines: Theory
Circles: PIC (Warp)
What is going on here?

$$r_{t2} = r_0 e^{-v_0^2/v_{r2}^2}$$

If $v_0 \ll v_{r2}$

$$r_{t2} \approx r_0$$  → Earlier, cold SC-dominated model approximate – slightly delays shock

If $v_0 \gg v_{r2}$

Initial velocity profile very important

→ Here: transforms Gaussian to uniform-like, i.e. it loses the shock
Analogous Gaussian

\[ v_0 = C \frac{r_0}{\sigma_r} \]

\[ v_{r2} = 1.4 \times 10^5 \frac{m}{s} \sqrt{1 - e^{-\frac{-r_0^2}{2\sigma_r^2}}} \]

Solid lines: Theory
Circles: PIC (Warp)

<table>
<thead>
<tr>
<th>Small velocity</th>
<th>Medium velocity</th>
<th>Large velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C = 0 )</td>
<td>( C = -10^4 \text{ m/s} )</td>
<td>( C = -10^5 \text{ m/s} )</td>
</tr>
</tbody>
</table>

Inward

\( C = 15 \text{ ns} \)
\( C = 17 \text{ ns} \)
\( C = 19 \text{ ns} \)
\( C = 21 \text{ ns} \)
\( C = 23 \text{ ns} \)
\( C = 25 \text{ ns} \)

\( C = 8 \text{ ns} \)
\( C = 9 \text{ ns} \)
\( C = 10 \text{ ns} \)
\( C = 11 \text{ ns} \)
\( C = 12 \text{ ns} \)
\( C = 13 \text{ ns} \)

\( C = 3.6 \text{ ns} \)
\( C = 4.0 \text{ ns} \)
\( C = 4.4 \text{ ns} \)
\( C = 5 \text{ ns} \)
What is going on here?

\[ r_{t2} = r_0 e^{-v_0^2/v_{r2}^2} \]

If \( v_0 \ll v_{r2} \)

\[ r_{t2} \approx r_0 \quad \rightarrow \text{Earlier, cold SC-dominated model still approximate – slightly moves shock earlier} \]

If \( v_0 \gg v_{r2} \)

Initial velocity profile very important

\( \rightarrow \) Here: linear velocity breaks laminar assumption in the middle of the bunch, so bunch bounces back faster in model than reality
Cylindrical Gaussian with spatially non-linear initial velocity

\[ v_0 = C \sqrt{1 - e^{\frac{r_0^2}{2\sigma_r^2}}} \]

\[ v_{r2} = 1.4 \times 10^5 \frac{m}{s} \sqrt{1 - e^{\frac{r_0^2}{2\sigma_r^2}}} \]

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Conclusions

- Surprisingly accurate self-consistent analytic model that predicts laminar density evolution
  - VERY fast
  - Able to predict through crossovers, i.e. focii
- Predicts when beam becomes non-laminar
  - Temperature?
- Physics captured by velocity scale, $v_{r2}$
- Spherical case in paper (similar)
- Expect pancake regime to be qualitatively similar to higher dimensions