Main and Fringe Field Computations for the Electrostatic Quadrupoles of the Muon g-2 Experiment Storage Ring

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Motivation

- Methods for measurement of anomalous MDM and EDM using storage rings rely on electrostatic particle optical elements.
- Accordingly, it is necessary to accurately model main and fringe fields of electrostatic elements.
- In particular, inaccurate treatment of fringe fields of electrostatic elements provides a mechanism for energy conservation violation.

Presentation Outline

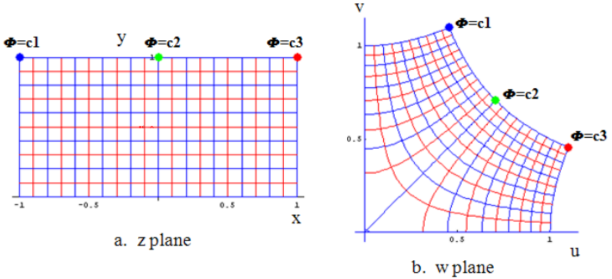
1. Fringe Fields of Electrostatic Deflectors

2. Main and Fringe Fields of the Muon g-2 Collaboration Quadrupole

Section 1

Fringe Fields of Electrostatic Deflectors

Conformal Mappings

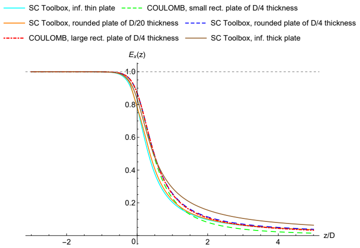


The Schwarz–Christoffel mapping $f(z) = \sqrt{z}$ maps the upper half-plane to the upper-right quadrant of the complex plane. (Image source: Kapania *et al.*)

- A conformal mapping (or conformal map) is a transformation $f : \mathbb{C} \to \mathbb{C}$ that is locally angle-preserving.
- Conformal mappings satisfy Cauchy-Riemann equations, which is useful for solving the Laplace equation.

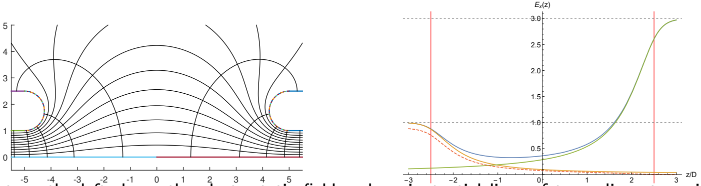
Fringe Fields of Semi-Infinite Capacitors

- Using conformal mappings, we obtained electrostatic field falloffs for semi-infinite capacitors with infinitely thin, infinitely thick, and finitely thick plates, including plates with rounded edges.
- There is a good agreement with fringe fields of several finite rectangular electrostatic capacitors obtained using a boundary element method (BEM) field solver.



Comparison of field falloffs of several semi-infinite capacitors computed in the *SC Toolbox* with field falloffs of two finite rectangular capacitors computed in *COULOMB*.

Fringe Fields of Two Adjacent Semi-Infinite Capacitors



The plot on the left shows the electrostatic field and equipotential lines of two adjacent semi-infinite capacitors with plates of $3D/4$ thickness, symmetric voltages, and rounded edges. The plot on the right shows the electrostatic field $E_{A\&B}(z)$ (blue) of two adjacent semi-infinite capacitors with plates of $D/2$ thickness and different voltages $V_A = 1$ and $V_B = 3$, individual fields $E_A(z)$ (orange) and $E_B(z)$ (green) of each capacitor as in empty space, and the difference $E_{A\&B}(z) - E_B(z)$ (dashed red) that would be equal to $E_A(z)$ without electrostatic induction.

We also modeled fringe fields of two adjacent semi-infinite capacitors with finitely thick plates and symmetric, antisymmetric, and different voltages.

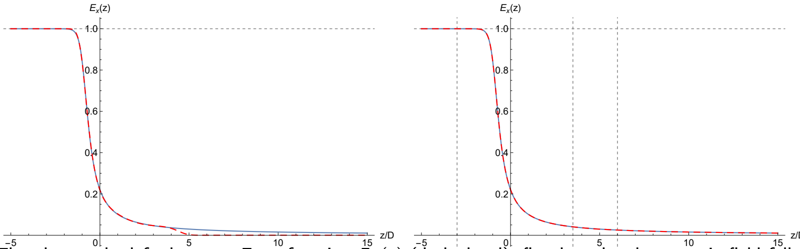
Accurate Fringe Fields Representation

- We found that the field falloff of an electrostatic deflector is slower than exponential.
- Enge functions of the form $F_N(z) = \frac{1}{1+\exp\left(\sum_{j=1}^{N} a_j \left(\frac{z}{D}\right)^{j-1}\right)}$ are not suitable for accurate modeling of the asymptotic behavior of said falloffs.
- We found that the alternative function

$$H(z) = \frac{1}{1+\exp\left[\sum_{j=1}^{N_1} a_j \left(\frac{z}{D}\right)^{j-1}\right]} \frac{1}{1+\exp\left[\left(\frac{z}{D}-c\right)^2\right]} + \\ + \frac{1}{\sum_{j=1}^{N_2} b_j \left(\frac{z}{D}\right)^{j-1}} \frac{1}{1+\exp\left[-\left(\frac{z}{D}-c\right)^2\right]}$$

models field falloffs of electrostatic deflectors accurately.

Accurate Fringe Fields Representation



The plot on the left shows an Enge function $F_5(z)$ (dashed red), fitted to the electrostatic field falloff $E_x(z)$ (solid blue) of a semi-infinite capacitor with infinitely thin plates. The plot on the right shows a function (dashed red) of the alternative form $H(z)$, fitted to $E_x(z)$ (solid blue) and enhanced in the interval $-3.5 \le z/D \le 6.5$ by adding a Fourier exponential series expansion of the difference $E_x(z) - H(z)$.

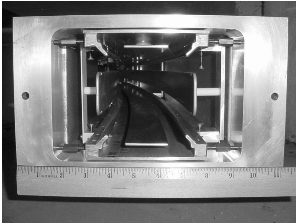
Section 2

Main and Fringe Fields of the Muon g-2 Collaboration Quadrupole

Main Field of the Muon g-2 Collaboration Quadrupole

The main field of an electrostatic element such as the Muon g-2 collaboration quadrupole may be obtained using the following method:

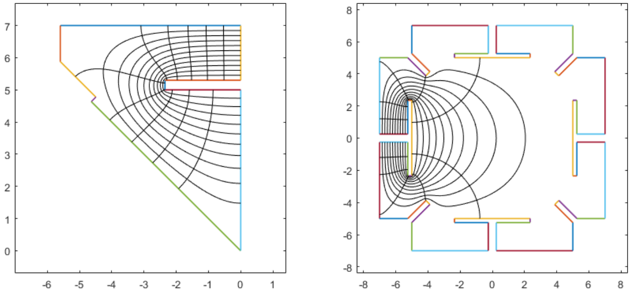
1. Calculate the electrostatic potential using conformal mapping methods with one plate at $1\ \mathrm{V}$ and the other Dirichlet boundary conditions (the remaining plates, the rectangular enclosure, and the trolley rails) of $0\ \mathrm{V}$.
2. Apply plate distance errors as perturbations to four copies of the potential, each copy corresponding to one plate at $1\ \mathrm{V}$ and the other Dirichlet boundary conditions of $0\ \mathrm{V}$.
3. Apply appropriate rotations to these four copies of the potential, scale the copies (e.g., by $\pm 2.4 \times 10^4$ or with mispowered values), and use their superposition.



The Muon g-2 collaboration quadrupole. (Image source: Semertzidis *et al.*)

The Muon ring at Fermilab. (Image source: FNAL.)

Nominal Symmetric and Non-Symmetric Models



The plots on the left and right show the polygonal model of the Muon g-2 collaboration quadrupole in the SM and NSM cases, respectively.

- We considered two polygonal models of the cross section: (1) the nominal case with symmetric voltages and no geometric asymmetries ("SM"), and (2) the general case of mispowered plates and geometric asymmetries ("NSM").
- In the former case, the polygonal model is simplified using reflection and rotation symmetries.

Conformal Mapping Derivative

In both cases, the derivative of the conformal mapping f from the canonical domain to the physical domain is

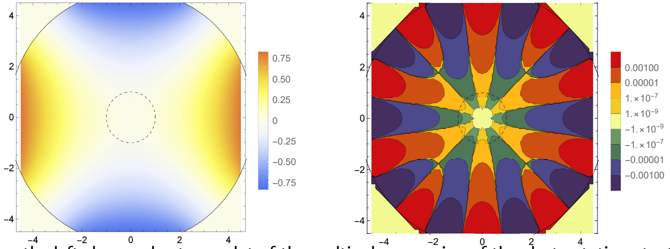
$$f'(z) = c\,\mathrm{cn}\,(z|m)\,\mathrm{dn}\,(z|m) \prod_{j=1}^{n} \left(\mathrm{sn}\,(z|m) - \mathrm{sn}\,(x_j + iy_j|m)\right)^{\alpha_j - 1},$$

where sn, cn, and dn are the Jacobi elliptic functions[1], K is the complete elliptic integral of the first kind[2], the parameters n and α were obtained from the polygonal model, and the parameters x, y, m, and c were found using the *SC Toolbox*.

[1] Definitions of the Jacobi elliptic functions can be found at `http://mathworld.wolfram.com/JacobiEllipticFunctions.html`.

[2] The complete elliptic integral of the first kind is defined at `http://mathworld.wolfram.com/CompleteEllipticIntegraloftheFirstKind.html`.

Multipole Terms

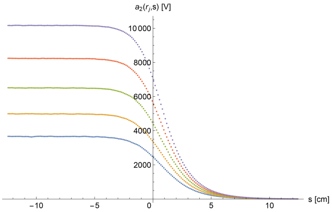


The plot on the left shows a heatmap plot of the multipole expansion of the electrostatic potential in the NSM case, up to order 24. The plot on the right shows a contour plot of the multipole expansion of the electrostatic potential in the NSM case, orders 3 to 24.

- We obtained the multipole expansion of the electrostatic potential using the differential-algebraic (DA) inverse of the conformal mapping, as well as using Fourier analysis applied to the conventional inverse.
- The conformal mappings method has the advantage of an analytic, fully Maxwellian formula and allows rapid recalculations with adjustments to the geometry and mispowered plates.
- The applicability of the conformal mapping method is limited by the crowding phenomenon; it can be expanded to more complex geometries using the cross ratios of the Delaunay triangulation (CRDT) algorithm.

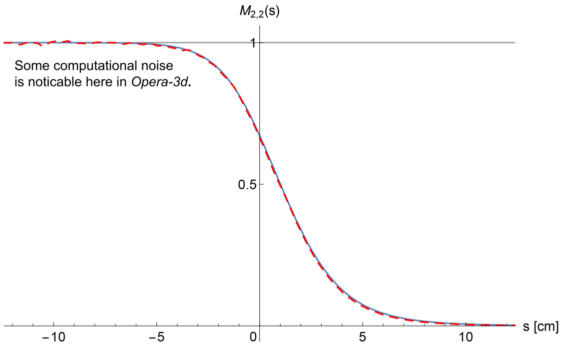
Fringe Field of the Muon *g*-2 Collaboration Quadrupole

- We obtained the quadrupole strength falloff and the EFB $z_{\mathrm{EFB}} = 1.2195\mathrm{cm}$ for the Muon *g*-2 collaboration quadrupole by calculating Fourier modes of its electrostatic potential at a set of radii in the transversal plane.
- The electrostatic potential data was obtained using a BEM field solver from a 3D model of the quadrupole.
- For a confirmatory comparison, we applied the same method of calculating multipole strengths to the electrostatic field data obtained for the Muon *g*-2 collaboration quadrupole using *Opera-3d*'s finite element method (FEM) field solver by Wanwei Wu (FNAL).



Falloffs of 2nd order Fourier modes $a_2(r_j)$ calculated at radii $r = 1.8, 2.1, 2.4, 2.7, 3.0$ cm from Wu's field data. Curves with larger magnitudes correspond to larger radii.

Results Based on Soltner–Valetov and Wu Field Data



The falloff of the multipole term $M_{2,2}$ agrees well between calculations based on Soltner–Valetov field data ($z_{\mathrm{EFB}} = 1.2195\ \mathrm{cm}$; solid blue) and field data by Wu ($z_{\mathrm{EFB}} = 1.1233\ \mathrm{cm}$; dashed red).

- The field falloffs and the EFBs obtained from Soltner–Valetov and Wu field data are in good agreement, and so are the tunes based on them.
- The quadrupole strength and the EFBs we obtained using this method explained the experimentally measured tunes, while simple estimates based on a linear model exhibited discrepancies up to 2%.

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Thank you!